A DISCUSSION OF THE CASES WHEN TWO QUADRATIC EQUATIONS INVOLVING TWO VARIABLES CAN BE SOLVED BY THE METHOD OF QUADRATICS.*

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1. The treatment of simultaneous quadratics in our elementary text-books is most unsatisfactory to the teacher. The author sometimes begins with the theorem, "The solution of a system of quadratic equations involving two variables in general requires the solution of a biquadratic." More often, no mention is made of the general theorem. Three cases are stated where special devices make possible the solution by quadratics. Then follows a set of problems, some under these three heads, many not. The pupil is left to use his ingenuity, with more or less suggestion from others, in solving them.

Chrystal says: "A moderate amount of practice in solving puzzles of this description is useful as a means of cultivating manipulative skill, but he (the student) should beware of wasting his time over what is, after all, merely a chapter of accidents."

The purpose of the following paper is to attempt to remove the problem from the category of a "chapter of accidents," or the realm of puzzles, and make it possible for the teacher, if not the pupil, to determine when the solution of such a system is possible by the method of quadratics. The conditions that the given equations can be solved entirely by the method of quadratics are obtained. In certain cases where these tests fail, it is still possible to obtainq uadratic factors, or a linear factor which will give a partial solution. The latter case is of especial interest, as indicated by the discussions in Vol. VI, pages 13-14, and Vol. VII, page 169, of THE AMERICAN MATHEMATICAL MONTHLY.

For the following work, Chrystal's *Text-book of Algebra*, Third Edition, Part I, pages 416-417; Burnside and Panton's *Theory of Equations*, Third Edition, pages 129-130; and Dr. K. L. Bauer's article in *Hoffmann's Zeitschrift*, 1874, page 317, have been found most helpful.

2. Let the given system of equations be represented by

$$ax^2+by^2+cxy+dx+ey+f=0\ldots\ldots(\mathrm{I})$$

$$a'x^2+b'y^2+c'xy+d'x+e'y+f'=0\ldots\ldots(\mathrm{II}).$$

Following Chrystal's suggestion as to method of solution, let

$$cy+d=p, \quad by^2+ey+f=q,$$

$$c'y+d'=p', \quad b'y^2+e'y+f'=q'.$$

Substituting these values in (I) and (II), we get

* Submitted through Professor Cajori and somewhat condensed by Editor Dickson.

$$ax^2+px+q=0, \quad a'x^2+p'x+q'=0.$$

Eliminating x^2, and finally, also x itself, we get

$$(aq'-a'q)^2-(ap'-a'p)(pq'-p'q)=0.$$

Substituting the values assumed for p and q,

$$[a(b'y^2+e'y+f')-a'(by^2+ey+f)]^2-\{[a(cy'+d')$$

$$-a'(cy+d)][(cy+d)(b'y^2+e'y+f')-(c'y+d')(by^2+ey+f)]\}=0 \text{....(III)}.$$

In expanding the last form the coefficients are such that determinants can be used to advantage. The determinant $\begin{vmatrix} a & b \\ a' & b' \end{vmatrix}$ or $ab'-a'b$ will for convenience be written in the form $[ab']$. The equation takes the form

$$Ay^4+By^3+Cy^2+Dy+E=0 \text{.....}(Q).$$

where

$$A=[ab']^2+[ac'][bc'],$$

$$B=2[ae'][ab']-[ac'][ce']+[ac'][bd']+[ad'][bc'],$$

$$C=[ae']^2+2[ab'][af']-[ac'][cf']-[ac'][de']-[ad'][ce']+[ad'][bd'],$$

$$D=2[ae'][af']-[ac'][df']-[ad'][cf']-[ad'][de'],$$

$$E=[af']^2-[ad'][df'].$$

3. If the quartic (Q) is separable into quadratic factors, the quartic may be said to be irreducible or reducible, according as the coefficients do or do not involve irrational numbers.

I. Irreducible quartic:

(1) $[(r+\sqrt{m})y^2+ny+p][(r-\sqrt{m})y^2+ny+p]=0.$

(2) $[my^2+(r+\sqrt{n})y+p][my^2+(r-\sqrt{n})y+p]=0.$

(3) $[my^2+ny+(r+\sqrt{p})][my^2+ny+(r-\sqrt{p})]=0.$

II. Reducible quartic:

(1) Solvable by method of quadratics. The same forms as I, if $\sqrt{m}$, $\sqrt{n}$, $\sqrt{p}$ are rational.

(2) Solvable by factoring of quartic, but not by the method of quadratics, pure and simple:

(a) $(my^2+ny+p)(m'y^2+n'y+p')=0.$

(b) $(my^3+ny+py+q)(m'y+q')=0.$

4. All equations of the first form I, (1), can be written

$$(ry^2+ny+p)^2-my^4=0, \text{ or } (Xy^2+\frac{D'}{2}y+1)^2-Yy^4=0\text{.....}(IV),$$

in which the absolute term is obtained by dividing both factors by p; $D'=D/E$, D and E as in (Q), X and Y to be found.

Expanding (IV) and comparing the coefficients with the coefficients of the same powers in (Q), after dividing (Q) by E,

$$A/E=X^2-Y, \quad D'X=B/E, \quad D'/4+2X=C/E.$$

Equating the values of X derived from the last two,

$$D^3+8BE^2=4CDE\text{.....}(J).$$

Therefore (J) is the condition that the quartic be capable of being resolved into factors of the first form.

5. All equations of the second form I, (2), can be written

$$(my^2+ry+p)^2+ny^2=0 \text{ or } [\sqrt{(a)}y^2+Xy+\sqrt{E})-Yy^2=0\text{.....}(V),$$

in which A and E are the same as in (Q), X and Y to be found. Expanding and comparing the coefficients with the coefficients of the same powers in (Q),

$$2\sqrt{a}\,X=B, \quad X^2+2\sqrt{(AE)}-Y=C, \quad 2X\sqrt{E}=D.$$

$\therefore B^2E=D^2A$.....(K).

Therefore (K) is the condition that the quartic (Q) shall be capable of being resolved into factors of the second form.

6. All equations of the third form can be written

$$(my^2+ny+r)^2-p^2=0 \text{ or } (y^2+\frac{B'}{2}y+X)^2-Y=0\text{.....}(VI),$$

in which $B'=B/A$, B and A as in (Q), X and Y to be found. Expanding and comparing coefficients with the coefficients of the like powers in (Q), after dividing by A,

$$2X+B'^2/4=C/A, \quad B'X=D/A, \quad X^2-Y=E/A.$$

$\therefore B^3+8A^2D=4ABC$.....($L$).

Therefore (L) is the condition that the quartic (Q) be capable of being resolved into factors of the third form.

The test (L) is the one obtained by Dr. K. L. Bauer in his article referred to in the introduction.

As an example under the case I of irreducible quartic, take

$$5y^2+2y-x-3=0, \quad x^2+6x+40y^2+16y+20=0.$$

Then (Q) becomes

$$25y^4+20y^3+44y^2+16y+11=0.$$

Trying successively the tests (J), (K), (L), the last one gives the identity

$$(20)^3+8.(25)^2.16=4.25.20.44.$$

Then $X=D/B=\frac{4}{5}$, $Y=X^2-E/A=\frac{1}{5}$.

$$[y^2+\tfrac{2}{5}y+\tfrac{4}{5}+\sqrt{\tfrac{1}{5}}][y^2+\tfrac{2}{5}y+\tfrac{4}{5}-\sqrt{\tfrac{1}{5}}]=0.$$

7. If (Q) is separable into rational factors, these factors may be both quadratic, or one linear and one cubic. If the factors are quadratic, the roots of the quartic may be all irrational, two irrational and two rational, or all rational. The general form for such a quartic is,

$$(my^2+ny+p)(m'y^2+n'y+p')=0 \text{.....(VII)}.$$

If the radical terms disappear in the three forms already considered, the resulting forms are special cases under (VII). If, therefore, the tests (J), (K), or (L) give a result in which $\sqrt{Y}$ is rational, the roots of the quartic may be obtained by the method used for the irreducible case.

8. Consider next the special devices that lead to the factoring of quartic, when the previous direct tests fail and the quartic cannot be solved by the method of quadratics.

If the quartic (Q) is separable into two quadratic factors, the general form (VII) has been given. One of these factors may be obtained as follows. Dividing (Q) by A, the coefficient of y^4, and transforming the resulting equation so as to remove fractional coefficients, if this be necessary, let the resulting quartic be

$$y^4+a_1y^3+a_2y^2+a_3y+a_4=0 \text{.....}(Q').$$

If (Q') is separable into quadratic factors, let

$$y^2+\alpha y+\beta \text{.....}(P)$$

be one of those factors. Dividing the first member of (Q') by this factor, the remainder is

$$[(a_3-a_2\beta+\alpha\beta)-(a_2\alpha-\alpha\beta-a_1\alpha^2+\alpha^3)]y+a_4-(a_2\beta-\beta^2-a_1\alpha\beta+\alpha^2\beta).....(R),$$

and the quotient is,

$$y^2+(a_1-\alpha)y+(a_2-\beta-a_1\alpha+\alpha^2).....(P').$$

If $(R)=0$, then (P) and (P') are the quadratic factors of (Q').

The two conditions that $(R)=0$, identically, are

$$(1)\ \ a_3-a_1\beta+\alpha\beta=a_2\alpha-a_1\alpha^2+\alpha^3-\alpha\beta,$$

$$(2)\ \ a_4=a_2\beta-\beta^2-a_1\alpha\beta+\alpha^2\beta.$$

From them

$$\frac{a_3-a_1\beta+2\alpha\beta}{\alpha}=a_2-a_1\alpha+\alpha_2,\ \ \frac{a_4}{\beta}+\beta=a_2-a_1\alpha+\alpha_2.$$

Equating the first members, and solving for α,

$$\alpha=(a_3\beta-a_1\beta^2)/(a_4-\beta^2).$$

β must be an integral factor of a_4. Therefore if a value for β can be found by factoring a_4, which will give the above expression for α integral, the factor (P) can be determined and second factor (P') readily found.

If the quartic is reducible, and one factor linear and one an irreducible cubic, the solution cannot be effected by the method of quadratics.

If however, (Q) be divided by A, and the resulting equation transformed to remove fractional coefficients, if necessary, the single real and rational root may be found by the Remainder Theorem. Thus one set of values for x and y in the original equations may be found.

There are some special cases under this general one that are worthy of consideration. Systems of quadratics of the form

$$x^2+y=a,\quad y^2+x=b,$$

have been of especial interest. (See end of §1).

It can readily be shown, by applying the tests (J), (K), and (L), that the resulting quartic does not come under any of the cases solvable directly by the method of quadratics. But sometimes by a special device we can find one set of values for x and y. This is the case when the quartic can be separated into one linear and one irreducible cubic factor. For instance, let the general form of such a system be

$$\left.\begin{array}{l}(ax+by+c)^2-d^2=e-(fx+gy+h)\\(fx+gy+h)^2-e^2=d-(ax+by+c)\end{array}\right\}.....(F).$$

That

$$ax+by+c=d,\quad fx+gy+h=e,$$

can be seen by inpection. If the given equations can be arranged as in (F), the solution as far as obtainable by this device, is readily completed.

To separate (I) and (II) as given in (F), extract the square roots of the first members of (I) and (II) as far as possible. If the remainder obtained in the first is the root obtained in the second, and the remainder in the second the root in the first, the equations may be written in the form (F).

A special case of the above general form is

$$x^2-d^2=e-y, \quad y^2-e^2=d-x,$$

in which $x=d$, $y=e$. See the references at end of §1.

Another of these special cases is the one in which the single real and rational root is a quartic can be obtained by the method next explained. The following two theorems will be used, taken from Eugen Netto's *Vorlesungen über Algebra*, Vol. I, page 56:

(1) If all the coefficients, c_1, $c_2 ..... c_n$ of the polynomial

$$f(y)=y^n+c_1y^{n-1}+.....+c_{n-1}y+c_n$$

are integral and divisible, without a remainer, by a prime number p, but c_n is not divisible by a higher power of p than the first, then $f(y)$ is irreducible.

(2) If all the coefficients, c_1, $c_2 ..... c_n$ of the polynomial

$$f(y)=y^n+c_1y^{n-1}+.....+c_{n-1}y+c_n$$

are divisible by a prime number p, but c_{n-1} is not divisible by a higher power of p than the first, $f(y)$ is either irreducible or is reducible to a factor of the first degree and an irreducible factor of the $(n-1)$th degree.

If the coefficients a_1, a_2, a_3, a_4 of the quartic (Q') are divisible by a prime number p, but a_3 is not divisible by a higher power of p than the first, and the quartic is reducible to a linear factor and an irreducible quartic, one root of the quartic, and one set of values of x and y can be ascertained without recourse to the regular algebraic solution. Let the cubic be

$$y^3+c_1y^2+c_2y+c_3=0, \quad c_1=p^nr, \quad c_2=p^mr, \quad c_3=pr_2.$$

If the factor $y+a$ is introduced,

$$y^4+(a+p^nr)y^3+(ap^nr+p^mr_1)y^2+(ap^mr_1+pr_2)y+apr_2=0.....(M).$$

$$\text{Let } a+p^nr=B, \quad ap^n+p^mr_1=C, \quad ap^mr_1+pr_2=D, \quad apr_2=E.$$

Then a system of quadratic equations giving the quartic (M) is

$$2y^2+By=x, \quad x^2+(4C-B^2)y^2+4Dy+4E=0.$$

9. There are certain forms of simultaneous quadratics which are readily recognized as solvable by the method of quadratics. It may be interesting to place these according to the preceding discussion. The classification given by Chrystal, Algebra, Part I, page 417, has been followed quite closely.

I. Two roots zero. The quartic (Q) assumes the form

$$Ay^4+By^3+Cy^2=0 \text{ or } (Ay^2+By+C)y^2=0.$$

It is evident that the conditions $D=0$, $E=0$ are satisfied if

$$ad'=a'd,\ \ af'=a'f,\ \ df'=d'f, \text{ or } a/a'=d/d'=f/f' \ldots\ldots (N).$$

But if $D=0$, $E=0$ tests (J) and (K) are satisfied. Therefore the system of quadratics is solvable by the method of quadratics.

II. Two roots infinity. The quartic (Q) assumes the form

$$Cy^2+Dy+E=0.$$

It is evident that a sufficient condition for the vanishing of A and B is

$$ab'=a'b,\quad ac'=a'c,\quad bc'=b'c, \text{ or } a/a'=b/b'=c/c'$$

But if $A=0$, $B=0$, the tests (K) and (L) are satisfied; therefore the quadratics are solvable as before.

III. The quartic contains only the even powers of y. The quartic (Q) assumes the form $ay^4+cy^2+E=0$. A sufficient condition for the vanishing of B and D is

$$d=0,\quad d'=0,\quad e=0,\quad e'=0.$$

But these are the conditions that (I) and (II) are homogeneous, and the tests (J), (K), and (L) are satisfied, so that two homogeneous equations and any other systems producing a quartic (Q) containing only the even powers of y, are solvable by the method of quadratics.

IV. If the quartic is a reciprocal equation of the form

$$(1)\ \ Ay^4+By^3+Cy^2+By+A=0,$$

then $A=E$, $B=D$. But if $A=E$,

$$\{ab'\}^2+\{ac'\}\{bc'\}=[\{af'\}^2-\{ad'\}\{df'\}],$$

a condition which is satisfied if $b=f$, $b'=f'$, $c=d$, $c'=d'$. If these conditions hold, $B=D$, so that the quartic is reciprocal.

But if $A=E$, $B=D$, the test (K) is satisfied. Therefore a system of quadratics producing a reciprocal quartic of the form (1) is solvable as before.

If the quartic assumes the reciprocal form

$$(2)\ Ay^4+By^3-By-A=0,$$

the above test (K) does not hold. But if the equation be divided by A, it is seen as in §8 that $a=0$, so that y^2-1 is a factor of the reciprocal equation. The remaining factor is readily found.

V. If the quadratic equations are symmetrical,

$$a=b,\quad d=e,\quad a'=b',\quad d'=e'.$$

Equations (I) and (II) then become, after division by a and a',

$$x^2+y^2+c_2xy+d_1x+d_1y+f_1=0\text{.....(I')},$$

$$x^2+y^2+c_1xy+d_2x+d_2y+f_2=0\text{.....(II')}.$$

Subtracting, $$c_3xy+d_3x+d_3y+f_3=0\text{.....(III')}.$$

Substituting $x=\mu+\nu$, $y=\mu-\nu$ in (I') and (III'),

$$a_1\mu^2+b_1\nu^2+2d_1\mu+f_1=0\text{.....(IV')},$$

$$c_3\mu^2-c_3\nu^2+2d_3\mu+f_3=0\text{.....(V')}.$$

From (IV') and (V'), ν^2 may be eliminated, and the resulting quadratic in μ readily solved. Or the values for a, a', etc., may be substituted in (III). In this case the terms in ν^3 and ν will vanish and the resulting quartic in ν comes under the special case III just discussed.